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2008 J. Phys.: Condens. Matter 20 395215

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Enhanced quantum tunnelling induced by disorder

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Received 9 May 2008

Published 1 September 2008

Online at stacks.iop.org/JPhysCM/20/395215

Abstract

We reconsider the problem of the enhancement of tunnelling of a quantum particle induced by disorder of a one-dimensional tunnel barrier of length L , using two different approximate analytic solutions of the invariant embedding equations of wave propagation for weak disorder. The two solutions are complementary for the detailed understanding of important aspects of numerical results on disorder-enhanced tunnelling obtained recently by Kim *et al* (2008 *Phys. Rev. B* **77** 024203). In particular, we derive analytically the scaled wavenumber (kL) threshold where disorder-enhanced tunnelling of an incident electron first occurs, as well as the rate of variation of the transmittance in the limit of vanishing disorder. Both quantities are in good agreement with the numerical results of Kim *et al*. Our non-perturbative solution of the invariant embedding equations allows us to show that the disorder enhances both the mean conductance and the mean resistance of the barrier.

Growing attention has been devoted in recent years to an intriguing phenomenon occurring when a quantum-mechanical particle of energy E impinges on a disordered one-dimensional tunnel barrier of mean height $V > E$. Indeed it was found that the transmission coefficient of the barrier is increased by the effect of a weak disorder [1–3]. This is surprising since one would expect that the additional scattering of an incoming electron by the disorder (e.g. random impurity potentials) would reduce the transmission rather than enhancing it.

The interest in the study of the transmission of quantum particles through disordered one-dimensional barriers [1–3] was triggered by an earlier study of transmission of scalar waves through disordered one-dimensional dielectric media [4]. In this case, the increase of the transmission coefficient with increasing disorder was first observed for frequencies of the incident wave lying in the gap of the band structure of the periodic medium in the absence of disorder [4]. The similarity between the quantum barrier and the dielectric medium problems comes from the fact that wave frequencies in the gap of the dielectric medium correspond to evanescent wave solutions of the wave equation, which are analogous to the evanescent states solutions of the Schrödinger equation for energies $E < V$ in the potential barrier problem.

Prior to the studies of transmission by disordered potential barriers in [1–3], the author developed an extensive analytic study of resistance and reflectance distributions in one-dimensional disordered tunnel barriers in the context of

electron localization [5, 6]. These studies, based on the invariant embedding method [7, 8], were motivated by the development of a fully probabilistic scaling theory of localization [9] in these systems where localization effects coexist with the familiar barrier penetration (tunnelling) effects [5, 6].

The purpose of this brief paper is first to apply the approximate solution for the reflection amplitude of a disordered barrier obtained in [5] for deriving analytic expressions for mean disorder-enhanced transmission coefficients and to compare the results with previous works [1–3]. In particular, we are interested in understanding an important new aspect of the transmission of a disordered barrier of length L revealed by the numerical results of Kim *et al* [3], namely the existence of a threshold value of the scaled particle wavenumber kL above which transmission is enhanced by the disorder while being reduced by the disorder below it.

On the other hand, it is found that the thresholds for enhanced transmission obtained numerically [3] vary significantly with the relative barrier height V/E . In order to understand this behaviour, it is necessary to develop a solution of the invariant embedding equations for arbitrary barriers $V/E > 1$, in parallel to the solution in [5] which is specific to the case $V/E = 2$. Such a solution based on perturbation theory for weak disorder is discussed below and applied to study the disorder-enhanced tunnelling. The relative merits of

the two types of solutions for describing the disorder-enhanced tunnelling are discussed towards the end of the paper.

On the other hand, we close by pointing out that the non-perturbative invariant embedding solution for a tunnelling barrier in [5], which does lead to the phenomenon of disorder-enhanced tunnelling, also leads to the exponential growth of resistance and conductance of a disordered barrier discussed earlier by Freilikher *et al* [1].

We consider an electron of energy $E = \hbar^2 k^2 / 2m$ (with units such that $\hbar = m = 1$) which impinges from the right on a random one-dimensional tunnel barrier

$$V(x) = V + v(x), \quad (1)$$

confined to the region $0 \leq x \leq L$. V denotes the mean of $V(x)$ and $v(x)$ is a weak Gaussian white noise:

$$\langle v(x)v(x') \rangle = \xi \delta(x - x'), \quad \langle v(x) \rangle = 0. \quad (2)$$

Outside the barrier the particle is described by the wavefunction

$$\psi(x) = e^{-ik(x-L)} + r(L) e^{ik(x-L)}, \quad x > L, \quad (3a)$$

$$\psi(x) = t(L) e^{-ikx}, \quad x < 0, \quad (3b)$$

where the complex reflection and transmission coefficient amplitudes $r(L) \equiv r$ and $t(L) \equiv t$ are determined by the invariant embedding equations [7]

$$ik \frac{dr(L)}{dL} = -2k^2 r(L) + V(L)(1 + r(L))^2, \quad (4)$$

$$ik \frac{dt(L)}{dL} = -k^2 t(L) + V(L)(1 + r(L))t(L). \quad (5)$$

In [5] we discussed a useful approximate solution of (4) valid within some energy interval around the value $E = V/2$ such that for the most typical values of (1) one has

$$2E - V \simeq v(L). \quad (6)$$

In this case the rhs of (4) is approximately $V(L)(1 + r(L))^2$ and the solution of (4) subject to $r(0) = 0$ is [5]

$$r(L) = -i \tanh \left[\frac{1}{k} \int_0^L dL' V(L') \right]. \quad (7)$$

By inserting (7) in (5) we obtain the corresponding exact solution for the amplitude transmission coefficient:

$$t(L) = \frac{e^{i(kL - \int_0^L dL' V(L'))}}{\cosh \left[\frac{1}{k} \int_0^L dL' V(L') \right]}, \quad (8)$$

which is valid for energies of the incident electron close to $V/2$. Note that (7), (8) verify probability conservation:

$$|r(L)|^2 + |t(L)|^2 = 1, \quad (9)$$

as required.

The solution (7) of the invariant embedding equation (4) for $E \simeq V/2$ has been further discussed by Haley and

Erdős [10] in the context of the resistance, $\rho(L)$, of the disordered barrier defined by the Landauer formula

$$\rho(L) = \frac{|r(L)|^2}{1 - |r(L)|^2}. \quad (10)$$

These authors also developed a more general result for the Landauer resistance of a disordered barrier valid for any value of V/E .

We first address the question of the disorder-enhanced tunnelling across the potential barrier (1) for weak disorder. More precisely, by expanding $|t(L)|^2$ in (8) to second order in $k^{-1} \int_0^L dL' v(L')$ assumed to be small and averaging over the disorder using (2), we get

$$\langle |t(L)|^2 \rangle = \frac{1}{\cosh^2 \frac{VL}{k}} \left[1 + \frac{\xi L}{k^2} \left(3 \tanh^2 \frac{VL}{k} - 1 \right) \right]. \quad (11)$$

It follows that the sign of the effect of the disorder on the transmittance of the barrier is given by the sign of the factor $3 \tanh^2(VL/k) - 1$. Thus, we find that for parameters such that $\exp(-\frac{2VL}{k}) < 2 - \sqrt{3}$, the transmission coefficient of the disordered barrier is enhanced by the effect of weak disorder. The threshold value

$$\frac{VL}{k} = \sqrt{V}L = kL = -\frac{1}{2} \ln(2 - \sqrt{3}) \simeq 0.659, \quad (12)$$

above which the mean transmission coefficient (11) increases with increasing disorder is in reasonable agreement with the critical value $kL \simeq 0.58$ at which the effect of the disorder in the transmittance changes sign in the results of figure 2 of [3]. Below this threshold value of kL the disorder reduces the transmittance while above it the disorder enhances it.

On the other hand, following Kim *et al*, we define the effective disorder parameter

$$g = \frac{\xi}{k^3}, \quad (13)$$

in terms of which we obtain from (11) (with $V = 2E = k^2$)

$$\frac{d\langle |t(L)|^2 \rangle}{dg} = \frac{kL}{\cosh^2 kL} (3 \tanh^2 kL - 1). \quad (14)$$

This defines the initial rate (i.e. near $g = 0$) of variation of the transmittance as a universal function of the scaled wavenumber kL , both above and below the threshold. The expression (14) may be compared with the slopes at the origin of the transmittances as a function of g in figure 2 of [3] for various values of kL . This comparison is shown in table 1, indicating a quite reasonable agreement between the two sets of results.

The enhancement of the transmittance of the tunnel barrier for weak disorder suggests, of course, a non-monotonic variation at larger disorder since beyond sufficiently large disorder the transmittance necessarily decreases to zero. This follows, for example, from the form of the typical transmission coefficient obtained from (8):

$$\begin{aligned} |t(L)|_{\text{typical}}^2 &= \left(\cosh^2 \left[\frac{1}{k} \int_0^L dL' V(L') \right] \right)^{-1} \\ &= 2 \left[1 + e^{\frac{2\xi L}{k^2}} \cosh \frac{2VL}{k} \right]^{-1}, \end{aligned} \quad (15)$$

Table 1. Initial ($g = 0$) slopes of transmittances as a function of disorder parameter g for various kL .

kL	Kim <i>et al</i> [3]	Equation (14)	Equation (28)
0.3	-0.194	-0.205	-0.169
0.577 (0.58)	0	-0.076	0
0.6	0.025	-0.057	0.019
0.659	—	0	0.068
1	0.414	0.311	0.25
1.5	0.557	0.395	—
3	0.206	0.058	—

which vanishes for $\xi \rightarrow \infty$. The result (15) is obtained by using the well-known formula [11]

$$\left\langle \exp \left[\pm a \int_0^L v(L') dL' \right] \right\rangle = \exp \left(\frac{a^2 \xi L}{2} \right), \quad (16)$$

for averages over Gaussian correlated variables defined by (2). The non-monotonic behaviour of the transmittance as a function of disorder is revealed in detail by the numerical calculations in [3, 4].

We now turn to the discussion of a new simple approximate solution of the invariant embedding equations, which in contrast to (7), (8), will be valid for arbitrary $E < V$. We first rewrite (4), (5) in terms of new amplitudes

$$q(L) = e^{-2ikL} r(L), \quad s(L) = e^{-ikL} t(L), \quad (17)$$

which lead to

$$ik \frac{dq(L)}{dL} = V(L) (e^{-ikL} + e^{ikL} q(L))^2, \quad (18)$$

$$ik \frac{ds(L)}{dL} = V(L) (1 + e^{2ikL} q(L)) s(L). \quad (19)$$

For $kL \ll 1$ these equations reduce approximately to

$$ik \frac{dq(L)}{dL} \simeq V(L) (1 + q(L))^2, \quad (20)$$

$$ik \frac{ds(L)}{dL} \simeq V(L) (1 + q(L)) s(L), \quad kL \ll 1, \quad (21)$$

and may be readily solved with the boundary conditions $q(0) = 0$ and $s(0) = 1$. We find

$$q(L) = \frac{1}{ik} \left(\int_0^L dL' V(L') \right) \left[1 - \frac{1}{ik} \int_0^L dL' V(L') \right]^{-1}, \quad (22)$$

$$s(L) = \left[1 - \frac{1}{ik} \int_0^L dL' V(L') \right]^{-1}, \quad kL \ll 1. \quad (23)$$

Restricting our attention to the transmittance, $|t(L)|^2 = |s(L)|^2$, we obtain from (1), (17) and (23)

$$|t(L)|^2 = \frac{1}{1 + \frac{1}{k^2} \left(VL + \int_0^L dL' V(L') \right)^2}. \quad (24)$$

By expanding (24) to second order in the weak disorder $v(L')$ in (1) and averaging the resulting expression using (2) we finally obtain

$$\langle |t(L)|^2 \rangle = \frac{1}{Q(L)} \left[1 + \frac{\xi L}{k^2 Q(L)} \left(\frac{4V^2 L^2}{k^2 Q(L)} - 1 \right) \right], \quad (25)$$

with

$$Q(L) = 1 + \left(\frac{VL}{k} \right)^2. \quad (26)$$

The expressions (25)–(26) show the existence of a wavenumber domain defined by $3 \left(\frac{VL}{k} \right)^2 > 1$, i.e.

$$kL > \frac{2}{\sqrt{3}} \frac{E}{V}, \quad (27)$$

where the transmittance is enhanced by the disorder. The critical scaled wavenumber threshold, $\frac{2}{\sqrt{3}} \frac{E}{V}$, may be readily compared with the critical thresholds for disorder-enhanced transmission obtained numerically by Kim *et al* [3] and displayed in their figures 3–5, successively for $V/E = 1.5$, $V/E = 2$ and $V/E = 3$. From these figures we obtain the critical values $kL \simeq 0.8$ ($V/E = 1.5$), $kL \simeq 0.58$ ($V/E = 2$) and $kL \simeq 0.4$ ($V/E = 3$), whose comparison with the results $kL = 0.77$ ($V/E = 1.5$), $kL = 0.577$ ($V/E = 2$) and $kL = 0.385$ ($V/E = 3$) obtained from (27) shows remarkable agreement.

Finally we discuss the initial slopes, $d\langle |t(L)|^2 \rangle / dg|_{g=0}$, of the mean transmittance (25) as a function of the disorder parameter (13), for various scaled incident wavenumbers kL . For the case $V/E = 2$ where numerical results for the transmittance are available in figure 2 of Kim *et al* [3], we get

$$\frac{d\langle |t(L)|^2 \rangle}{dg} = \frac{kL(3(kL)^2 - 1)}{(1 + (kL)^2)^3}, \quad kL \ll 1. \quad (28)$$

As shown in table 1, the agreement between the numerical results obtained from (28) with those inferred from the results of figure 2 of Kim *et al* [3] for various kL is again quite reasonable.

In conclusion, we have presented two complementary mathematical treatments demonstrating the existence of scaled wavenumber thresholds for the appearance of disorder-enhanced tunnelling of an electron and allowing us furthermore to calculate analytically the initial rate of variation of transmittance with the disorder as a function of the incident wavenumber. Our results are in good agreement with extensive numerical calculations of Kim *et al* [3].

We close with a brief comparison of our results for the special energy (6) for large L with the results of Freilikher *et al* [1] indicating that a weak disorder increases both the mean resistance and the mean conductance of the tunnel barrier. The resistance (ρ) of the barrier defined by the four-probe Landauer formula is

$$\begin{aligned} \rho &= \frac{|r(L)|^2}{|t(L)|^2}, \\ &= \sinh^2 \left[\frac{1}{k} \int_0^L dL' V(L') \right], \end{aligned} \quad (29)$$

using (7)–(9). For the conductance, g , the following two-probe formula:

$$g = |t(L)|^2, \quad (30)$$

is to be preferred, following well-known arguments [12].

From (2), (16) and (29) we obtain at long lengths such that $VL/k \gg 1$ (with $T_0(L) = 4 \exp(-2VL/k)$)

$$\langle \rho \rangle \simeq \frac{1}{T_0(L)} e^{\frac{2\epsilon L}{k^2}}, \quad (31)$$

and from (8) and (30)

$$\langle g \rangle = \langle |t(L)|^2 \rangle \simeq T_0(L) e^{\frac{2\epsilon L}{k^2}}. \quad (32)$$

These expressions are similar to the results obtained earlier by Freilikher *et al* [1], using a different method. Freilikher *et al* [1] have given an interesting interpretation of the simultaneous enhancement of the mean resistance and of the mean conductance by the disorder. This has to do with the fact that the resistance (29) and the conductance (30) of a disordered barrier are not self-averaging quantities [5, 12]. Therefore, the realizations of the random barrier which are responsible for the increase of the mean resistance and of the mean conductance are not the most typical ones. Under these conditions the analysis of Freilikher *et al* [1] suggests that the mean resistance and the mean conductance are influenced by different atypical realizations of the random tunnel barrier, in such a way that both quantities are increased.

As a final remark, we highlight the advantage of the approximate analytic solution (7), (8) of the invariant embedding equations at the energy of half the tunnelling barrier

height, in the context of the present work. On the one hand, this solution has enabled us to study the disorder-enhanced transmittance at wavenumbers larger than those which are accessible by means of the perturbation analysis discussed above (see, in particular, table 1), which requires $kL \ll 1$. On the other hand, its non-perturbative character is crucial for demonstrating the exponential growth of both the resistance and the conductance shown previously in [1].

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